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ON THE SYMBOLIC REPRESENTATION OF QUOTIENTIAL COEFFICIENTS OF THE SECOND ORDER.*

By ROBERT E. MORITZ, Ph. D., University of Washington.

The quotiential coefficient of a function of x, $y=f(x)$, has been defined by the expression†

$$\lim_{k=1} \log_k \frac{f(kx)}{f(x)}. \tag{1}$$

If we denote this limit by qy/qx, it may be shown that

$$\frac{qy}{qx} = \frac{x}{y} \cdot \frac{dy}{dx}, \tag{2}$$

and, for our present purpose, we may look upon (2) as the defining equation of the quotiential coefficient‡ of a function of x.

With the aid of this equation we derive immediately the following table of quotientation formulae:

*Read before the San Francisco Section of the American Mathematical Society.

†*American Journal of Mathematics*, Vol. XXIV, p. 265.

‡The term quotiential coefficient originates from the expression (1) which defines it as the limit of the logarithm of a quotient just as the differential coefficient is defined as the limit of a quotient of differences.

$$\frac{qa}{qx}=0,\ a\neq 0;\ \frac{qa}{qx}=\frac{0}{0},\ a=0; \tag{3_1}$$

$$\frac{qax^n}{qx}=n,\ \frac{qx}{qx}=1; \tag{3_2}$$

$$\frac{qa^x}{qx}=x\log a; \tag{3_3}$$

$$q\frac{\log x}{qx}=\frac{1}{\log x}. \tag{3_4}$$

If $y=u\pm v\pm w\pm$ etc., u, v, w, ..., being functions of x,

$$y\frac{qy}{qx}=u\frac{qu}{qx}\pm v\frac{qv}{qx}\pm w\frac{qw}{qx}\pm \text{ etc.} \tag{3_5}$$

If $y=u\times\div v\times\div w\times\div$ etc.,*

$$\frac{qy}{qx}=\frac{qu}{qx}\pm\frac{qv}{qx}\pm\frac{qw}{qx}\pm \text{ etc.} \tag{3_6}$$

$$\frac{qu^n}{qx}=n\frac{qu}{qx}. \tag{3_7}$$

If $u=f(z)$ and $z=\phi(x)$,

$$\frac{qu}{qx}=\frac{qu}{qz}\cdot\frac{qz}{qx}, \tag{3_8}$$

$$\frac{qu}{qx}=1/\frac{qx}{qu}. \tag{3_9}$$

If y is a function of n dependent variables, $y=F(u_1, u_2, \ldots)$, where $u_1=\phi_1(x)$, $u_2=\phi_2(x)$, etc.

$$\frac{qy}{qx}=\left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx}+\left(\frac{qy}{qu_2}\right)\frac{qu_2}{qx}+\text{ etc.}, \tag{4}$$

the brackets being used in the Eulerian sense to denote partial variation of the dependent variables, *i. e.*,

$$\left(\frac{qy}{qu_1}\right)=\frac{u_1}{y}\frac{\partial y}{\partial u_1},\ \text{etc.}$$

*$\times\div$ is used for the multiplication sign, $\times$, with dot in each of the vertical angles.

The proof of (4) is as follows:

$$\frac{dy}{dx} = \frac{\partial y}{\partial u_1}\frac{du_1}{dx} + \frac{\partial y}{\partial u_2}\frac{du_2}{dx} + \text{etc.},$$

hence,

$$\frac{qy}{qx} = \frac{x}{y}\frac{dy}{dx} = \frac{u_1}{y}\frac{\partial y}{\partial u_1}\cdot\frac{x}{u_1}\frac{dy}{dx} + \frac{u_2}{y}\frac{\partial y}{\partial u_2}\cdot\frac{x}{u_2}\frac{dy}{dx} + \text{etc.}$$

$$= \left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx} + \left(\frac{qy}{qu_2}\right)\frac{qu_2}{qx} + \text{etc.}$$

gy/gx is called the total quotiential coefficient of y with respect to x, and (qy/qu_1) the partial quotiential coefficient of y with respect to x.

Of course, all the preceding formulae may also be derived without a knowledge of differentiation by applying equation (1).

A partial quotiential coefficient of a partial quotiential coefficient we call a second partial quotiential coefficient, in symbols

$$\left[\frac{q\left(\frac{qy}{qu_1}\right)}{qu_2}\right] = \frac{q^2y}{qu_2\,qu_1}.$$

But unlike differentiation, partial quotientiation is not commutative, so that the order of the variables must be carefully attended to. We shall follow the order observed by American writers on differentiation, namely,

$$\left(\frac{q^2y}{qu_2\ qu_1}\right) = \left[\frac{q\left(\frac{qy}{qu_1}\right)}{qu_2}\right].$$

THEOREM. *If y is a function of two variables, u and v, then*

$$\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\ qv}\right) = \left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qv\ qu}\right). \tag{5}$$

Proof. By definition,

$$\left(\frac{q^2y}{qu\ qv}\right) = \frac{u}{\left(\frac{qy}{qv}\right)}\cdot\frac{\partial}{\partial u}\left(\frac{v}{y}\cdot\frac{\partial y}{\partial v}\right) = u\left(\frac{\partial^2 y}{\partial u\ \partial v} - \frac{1}{y}\cdot\frac{\partial y}{\partial v}\cdot\frac{\partial y}{\partial u}\right)\Big/\frac{\partial y}{\partial v},$$

$$\left(\frac{q^2y}{qv\ qu}\right) = \frac{v}{\left(\frac{qy}{qu}\right)}\cdot\frac{\partial}{\partial v}\left(\frac{u}{y}\cdot\frac{\partial y}{\partial u}\right) = v\left(\frac{\partial^2 y}{\partial v\ \partial u} - \frac{1}{y}\cdot\frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v}\right)\Big/\frac{\partial y}{\partial u},$$

whence,

$$\left(\frac{q^2y}{qu\,qv}\right):\left(\frac{q^2y}{qv\,qu}\right)=\frac{u}{y}\cdot\frac{\partial y}{\partial u}:\frac{v}{y}\cdot\frac{\partial y}{\partial v}=\left(\frac{qy}{qu}\right):\left(\frac{qy}{qv}\right)$$

which states the theorem to be proved in the form of a proportion.

Let us next quotientiate, with respect to x, according to our rules (3_1)—(3_9), each side of the expression

$$\frac{qy}{qx}=\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}.$$

After clearing of fractions there results

$$\frac{q^2y}{qx^2}\,\frac{qy}{qx}=\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)\frac{\overline{qu}^{\,2}}{qx}+\left[\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qv\,qu}\right)+\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu\,qv}\right)\right]\frac{qu}{qx}\,\frac{qv}{qx}$$

$$+\left(\frac{qy}{qv}\right)\frac{q^2u}{qv^2}\frac{\overline{qv}^{\,2}}{qx}+\left(\frac{qy}{qu}\right)\frac{qu}{qx}\,\frac{q^2u}{qx^2}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\,\frac{q^2v}{qx^2}. \qquad (6)$$

By applying (5) the expression within the brackets reduces to $2\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\,qv}\right)$

The resulting expression bears a striking resemblance to the corresponding expression for the differential quotient of the second order.

The first three terms on the right are now

$$\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)\frac{\overline{qu}^{\,2}}{qx}+2\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\,qv}\right)\frac{qu}{qx}\,\frac{qv}{qx}+\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qv^2}\right)\frac{\overline{qv}^{\,2}}{qx},$$

which, by dropping the partial quotiential coefficients of the first order and replacing at the same time q^2y by $\overline{qy}^{\,2}$, assumes the form of a perfect square, namely,

$$\left(\frac{qy}{qu}\right)^2\frac{\overline{qu}^{\,2}}{qx}+2\left(\frac{qy}{qu}\right)\left(\frac{qy}{qv}\right)\frac{qu}{qx}\,\frac{qv}{qx}+\left(\frac{qy}{qv}\right)^2\frac{\overline{qv}^{\,2}}{qx}=\left[\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\right]^2,$$

so that (6) may be written symbolically,

$$\frac{q^2y}{qx^2}\,\frac{qy}{qx}=\left[\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\right]^{(2)}+\left(\frac{qy}{qu}\right)\frac{qu}{qx}\,\frac{q^2u}{qx^2}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\,\frac{q^2v}{qx^2} \qquad (7)$$

where the exponent in parenthesis signifies, that the expression to which it is attached is to be squared, and after squaring, $\left(\frac{qy}{qu}\right)^2$ is to be replaced by $\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)$, $\left(\frac{qy}{qu}\right)\left(\frac{qy}{qv}\right)$ by $\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\,qv}\right)$, and $\left(\frac{qy}{qv}\right)^2$ by $\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qv^2}\right)$.

When $u=ax^m$, $v=bx^n$, formula (7) reduces to a remarkably simple from. For in that case $\frac{qu}{qx}=m$, $\frac{qv}{qx}=n$, $\frac{q^2u}{qx^2}=0$, $\frac{q^2v}{qx^2}=0$, hence all terms on the right of (7) outside of the bracket vanish, and we have

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\left(\frac{qy}{qu}\right)m+\left(\frac{qy}{qv}\right)n\right]^{(2)}. \tag{8}$$

Formulas (6), (7), and (8) may be at once extended to functions of any number of dependent variables. Let the the dependent variables be u_1, u_2, ..., u_n, then from (4),

$$\frac{qy}{qx}=\sum_{i=1}^{i=n}\left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx}.$$

Quotientiating again and remembering that generally

$$\left(\frac{qy}{qu_i}\right)\left(\frac{q^2y}{qu_k\,qu_i}\right)=\left(\frac{qy}{qu_k}\right)\left(\frac{q^2y}{qu_i\,qu_k}\right),\ i,\ k=1,\ \ldots,\ n,$$

we obtain

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\sum_{i=1}^{i=n}\left(\frac{qy}{qu_i}\right)\left(\frac{q^2y}{qu_i^2}\right)\frac{\overline{qu}_i^2}{qx}+2\sum_{\substack{k=2\\ i<k}}^{k=n}\left(\frac{qy}{qu_k}\right)\left(\frac{q^2y}{qu_i\,qu_k}\right)\frac{qu_i}{qx}\frac{qu_k}{qx}$$

$$+\sum_{i=1}^{i=r}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2}, \tag{9}$$

or in symbolic representation,

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\sum_{i=1}^{i=n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\right]^{(2)}+\sum_{i=1}^{i=r}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2}. \tag{10}$$

If, moreover, each of the functions is of the form $u_i=a_ix^{m_i}$, the second sum in expression (10) vanishes, $qu_i/qx=m_i$, and the expression reduces to

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\sum_{i=1}^{i=n}m_i\left(\frac{qy}{qu_i}\right)\right]^{(2)}. \tag{11}$$

SEVERAL HISTORIC PROBLEMS WHICH HAVE NOT YET BEEN SOLVED.

By DR. G. A. MILLER.

Goldbach's theorem affirms that every even number is the sum of two primes, unity being regarded as a prime number. For instance, $2=1+1$, $4=1+3=2+2$, $6=1+5=3+3$, $8=1+7=3+5$, $10=3+7=5+5$, $12=1+11=5+7$, etc. Although this theorem has been known for more than one hundred and sixty years* yet no one has succeeded in either proving or disproving it. Many attempts have been made, especially in recent years, but these have only served to make it more probable that the theorem is universally true. In two of the recent papers† an attempt is made to find some approxmate laws of increase of the number of ways in which an even number can be resolved into the sum of two primes when this even number becomes large. In the latter of these, the author reaches the conclusion that every even number which exceeds 50,000 is the sum of at least 330 different pairs of primes. It should, however, be emphasized that this result is not proved and that it still appears possible that some even number may be found which is not the sum of two primes.

Another theorem which remains unproved and has a still more extensive history than the one noted in the preceding paragraph is due to Fermat and is known under various names. Two of these are, *the last theorem of Fermat* and *the great theorem of Fermat.* It affirms that the equation

$$x^n+y^n=z^n$$

is not satisfied by any integral value of x, y, z whenever $n>2$; and it is equivalent to the theorem that the difference of the nth power of two rational fractions is never unity whenever $n>2$. The Greeks studied this equation for $n=2$ in connection with the rational right triangle. In his edition of Diophantus, Fermat states on the margin of a page that he has found a wonderful proof for the assertion that $x^n+y^n=z^n$ is not satisfied for integral values of x, y, z when $n>2$ but that the margin is too small to contain the proof. A large number of attempts have since been made but no one has been entirely successful although Kummer succeeded in proving it for an infinite number of special values of n. While it is very likely that the theorem is universally true yet it has not been *proved* impossible to find three integral values which satisfy the given equation for large values of n.

In Ball's History of Mathematics, third edition, page 306, the statement is made that it is not known whether there are any values of n to

*It is contained in letters written by Goldbach and Euler in 1742.

†Haussner, Jahresbericht der Deutschen Mathematiker-Vereinigung, Vol. 5 (1896), p. 62; Ripert, L' Intermediaire des Mathematiciens, Vol. 10 (1903), p. 78.

which Kummer's proof does not apply. This is incorrect as such values are not only known but it is also known that the number of such values increases as n increases* so that an infinite number of values of n do not come under Kummer's proof. Unfortunately Ball's History contains a large number of other inaccurate statements.†

As a third unsolved problem with a long history we may mention that no one has yet either proved or disproved the existence of an odd perfect number. A perfect number is one which like $6=1+2+3$ or $28=1+2+4+7+14$ is equal to the sum of its divisors. Euclid observed that $2^{p-1}(2^p-1)$ is a perfect number whenever 2^p-1 is a prime, and it has since been proved that every even perfect number is of this form. Nine such numbers are known. They increase so rapidly with p that their determination for large values of p is very laborious. Descartes thought it possible that odd perfect numbers exist yet none have been found although the subject has received a great deal of attention.

The questions mentioned above relate to properties of natural numbers that are easily understood. A similar problem which has a less extensive history is, whether there is an infinite number of pairs of primes such that the difference of the numbers which constitute a pair is 2. In other words, whether there is no largest prime such that this prime increased by 2 is again a prime. Many similar questions present themselves in number theory, where the very simple and the very difficult theorems seem to lie side by side without exhibiting any external evidence in regard to the class to which they belong.

A problem which has a much less extensive history than the preceding and which can probably be solved more readily is the determination of a six times transitive function which is neither symmetric nor alternating. This is equivalent to the determination of a six-fold transitive group which is neither alternating nor symmetric. In 1861, Mathien published‡ a five-fold transitive function on twelve letters and announced a similar function on twenty-four letters, which he afterwards explained more fully. Although more than forty years have passed since these discoveries no one has extended these results so that we are still ignorant of any six- or seven-fold transitive functions that are neither alternating nor symmetric. The more general question of determining a limit of transitivity for non-alternating and non-symmetric functions of a given degree has received considerable attention during this period and much progress has been made along this line.

One of the most beautiful theorems due to Lagrange affirms that every algebraic number of the second degree may be written in the form of a periodic continued fraction and that every periodic continued fraction is equal to an algebraic number of the second degree. The great importance

*Cf. Kronecker, Vorlesungen ueber Zahlentheorie, 1901, p. 23.

†Cf. On Ball's History of Mathematics, THE AMERICAN MATHEMATICAL MONTHLY, Vol. 9 (1902), p. 280.

‡Mathien, Journal de Mathematiques, Vol. 6 (1861), p. 241.

of this result led men like Jacobi and Hermite to make efforts to prove a similar theorem with respect to algebraic numbers of the third degree but their efforts were not crowned with success. Comparatively little progress has been made towards useful criteria to determine the degree of algebraic numbers, or even towards determining whether a given number is algebraic or transcendental. One of the problems suggested by Hilbert at the International Congress of Mathematicians held at Paris in 1900 is to determine whether a^p, the base being algebraic and the exponent an irrational algebraic number, always represents a transcendental or at least an irrational number.

It should not be inferred that the preceding problems are suggested as very suitable fields for the young investigator. The main object in stating them is to point out to those who may not have good library facilities that some problems relating to very elementary matters still remain unsolved, and, if possible, to encourage some one to acquaint himself with congeniel fields of study where much remains to be done. A clear understanding of the real nature of unsolved historic problems is of great importance to the stvdent since results bearing on such problems are of especial interest. Many problems of this kind are noted from time to time in *L' Intermediaire des Mathématiciens* published by Gauthier-Villars, and a set of about twenty very fundamental ones were given by Hilbert at the Congress mentioned in the preceding paragraph. These have been published in the *Bulletin of the American Mathematical Society*, Vol. 8 (1902), p. 437.

NOTE ON A RECENT PROBLEM IN THE AMERICAN MATHEMATICAL MONTHLY.

By R. D. CARMICHAEL, Anniston, Alabama.

The object of this note is to state in a somewhat more general form a proposition in number theory demonstrated on pages 155-156 of Volume XIII of the MONTHLY.

Given that $P^{\delta a}-R^{\delta a}$ *is divisible by* $^{\delta a}$, *the necessary and sufficient conditions that the expression* $\dfrac{P^a - R^a}{\delta a(P-R)}$ *shall be integral are:* (1) ^{a} *must be divisible by e, the least integer such that* P^e-R^e *is divisible by* ^{a_k}, *where for* ^{a_k} *is taken in turn the various prime factors of* ^{a} *not dividing* $P-R$; (2) $^\delta$ *is any divisor of* $(P^a - R^a)/a(P-R)$.

The proof is practically identical with that given in the MONTHLY (*l. c.*), except in showing that $^\delta$ and $1-R$ have no common factor. If $^{\delta_1}$ is such a prime factor, it becomes necessary in the present case to modify the proof that $^{\delta_1}$ must be a factor of ^{a}. This is shown as follows:

If δ_1 is contained in $P-R$, then $P=R+\delta_1 n^\lambda$, where λ i some integer and $n \gtreqless 1$. Hence,

$$P^{\delta_1}=R^{\delta_1}+\delta_1{}^{n+1}\lambda+\ldots \equiv R^{\delta_1} \pmod{\delta_1{}^{n+1}}.$$

Hence, $P^{\delta_1}-R^{\delta_1}$ is divisible by $\delta_1{}^{n+1}$. But $P^\alpha - R^\alpha$ is divisible by $\delta_1{}^{n+1}$. Now $P^{\delta_1}-R^{\delta_1}$ is the lowest number of this form which is divisible by $\delta_1{}^{n+1}$. Hence δ_1 is a factor of α.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

273. Proposed by THEODORE L. DE LAND, Treasury Department, Washington, D. C.

Three ingots of the precious metals were received at the Mint for assay, where it was found as follows: That in 3 grains of the first ingot and 2 grains of the second the gold was 3 times the silver; that in 2 grains of the first and 6 grains of the third the gold was 8 times the copper; that in 2 grains of the second and 3 grains of the third the silver was 5 times the copper; that in 1 grain of the first, 2 grains of the second, and 3 grains of the third the gold was 2 times the silver; that in 1 grain each of the first and second ingots there were 11 parts of gold to 5 parts of silver; and that 6 grains of the first, 5 grains of the second, and 2 grains of the third on being assayed proved to be 17 carats gold fine. There was no trace of any other metal in the ingots.

Required: The theoretical analysis of each of three ingots.

Solution by the PROPOSER.

Let x=the fraction of gold in a grain of the first ingot;
y=the fraction of silver in a grain of the first ingot;
$1-(x+y)$=the fraction of copper in a grain of the first ingot; and
z=the fraction of gold in a grain of the second ingot;
u=the fraction of silver in a grain of the second ingot;
$1-(z+u)$=the fraction of copper in a grain of the second ingot; and
v=the fraction of gold in a grain of the third ingot;
w=the fraction of silver in a grain of the third ingot;
$1-(v+w)$=the fraction of copper in a grain of the third ingot.

There are six unknown quantities and six conditions in the solution and problem which may be equated as follows:

First, $3x+2z=3[3y+2u]\ldots(1)$;
Second, $2x+6v=8\{2[1-(x+y)]+6[1-(v+w)]\}\ldots(2)$;
Third, $2u+3w=5\{2[1-(z+u)]+3[1-(v+w)]\}\ldots(3)$;
Fourth, $x+2z+3v=2(y+2u+3w)\ldots(4)$;
Fifth, $x+z : y+u :: 11 : 5\ldots(5)$; and
Sixth, $6x+5z+2v : 13 :: 17 : 24\ldots(6)$.

By elimination, we have $x=1$, $y=0$, $1-(x+y)=0$, $z=\frac{3}{8}$, $u=\frac{5}{8}$, $1-(z+u)=0$, $v=\frac{9}{24}$, $w=\frac{5}{24}$, and $1-(v+w)=\frac{1}{8}$.

We interpret these results as follows: That the first ingot was pure gold; that the second ingot was 9-carat gold, and 15-carat silver; and that the third ingot was 16-carat gold, 5-carat silver, and 3-carat copper.

Also solved in the same manner by G. B. M. Zerr, S. A. Corey, L. E. Newcomb, and G. W. Greenwood.

274. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find the limit of $\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\cdot\frac{7^2+1}{7^2-1}\cdot\frac{11^2+1}{11^2-1}\cdots$ where the squared numbers are the natural odd *primes* in order.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va., and J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Putting the expression in the form

$$\frac{(1+1/3^2)(1+1/5^2)(1+1/7^2)\ldots}{(1-1/3^2)(1-1/5^2)(1-1/7^2)\ldots}=\frac{s}{s_1}$$

and remembering that $(1+1/2^2)s=\frac{15}{\pi^2}$ and $(1-1/2^2)s_1=\frac{6}{\pi^2}$, (pp. 133-134, Vol. V, No. 5), we have $s/s_1=1\frac{1}{2}$.

C. N. Schmall gives the following arithmetical solution of 269. When the boats first meet, combined distance traveled is equal to width of river; when they meet for the second time the distance traveled is equal to three times the width of river and that each boat has gone three times as far as when they first met. Hence one has gone 3×720 yards$=$ 2160 yards and has made one trip and 400 yards of the return trip. Hence, width of river $=$2160 yards$-$400 yards$=$1760 yards$=$1 mile.

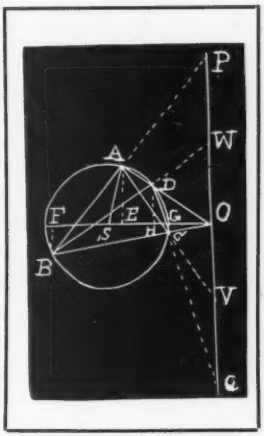
GEOMETRY.

302. Proposed by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Through a given point within a circle draw any two chords, also a radius and a secant perpendicular to the radius. Let the extremities of the chords be taken as the vertices of a quadrilateral. Show that the sides of the quadrilateral, produced when necessary, cut the secant in points equidistant, in pairs, from the given point. [A proof by Euclidean geometry is preferred, as the problem was originally given to a high school class.] Must the given points be within the circle?

Solution by the PROPOSER.

The problem holds for the complete quadrilateral and for the given point taken inside or outside of the circle. Of the three pairs of intercepts on the secant, one is null, corresponding to the given point. The following proof is for the case of the given point outside of the circle and the intercepts by the *diagonals* upon the secant. The method is applicable to all cases.



In the diagram, O is the given point, A, B, C, D are the vertices of the quadrilateral, and E, F, G, H are the projections of the respective vertices upon the diameter determined by O and the center of the circle at S. Let a be the radius of the circle and let SO be b. To prove that the diagonals AC and BD, produced, cut off equal distances OV and OW on the perpendicular to SO at O. From the similar triangles OAC and OBD,

$$\frac{\triangle OAC}{\triangle OBD}=\frac{OA^2}{OB^2}=\frac{OC^2}{OD^2}\ldots(1).$$

Since $\triangle OAC=\triangle OAV-\triangle OCV$, $\triangle OBD=\triangle OBW-\triangle ODW$, and using OV and OW as bases, it follows that

$$\frac{\triangle OAC}{\triangle OBD}=\frac{OV(OE-OG)}{OW(OF-OH)}\ldots(2).$$

From (1) and (2),

$$\frac{OV}{OW}=\frac{OA^2}{OB^2}\cdot\frac{OF-OH}{OE-OG}\ldots(3).$$

Noticing that OE and OG are the projections of OA and OC, the triangles SOA and SOC give, $a^2=b^2+OA^2-2b.OE$, $a^2=b^2+OC^2-2b.OG$.

$$\left.\begin{aligned}\therefore OE-OG&=\frac{OA^2-OC^2}{2b}\\ \text{Similarly, } OF-OH&=\frac{OB^2-OD^2}{2b}\end{aligned}\right\}\ldots(4).$$

From (3) and (4), using also the latter part of (1),

$$\frac{OV}{OW}=\frac{OA^2}{OB^2}\cdot\frac{OB^2-OD^2}{OA^2-OC^2}=\frac{OA^2.OB^2-OA^2.OD^2}{OB^2.OA^2-OB^2.OC^2}=1.$$

Also solved by C. N. Schmall, A H. Holmes, and L. E. Newcomb. Mr. Newcomb's demonstration was exhaustive, covering the cases when the point is within and without the circle. Our space is too limited to publish his demonstration.

CALCULUS.

230. Proposed by C. N. SCHMALL, College of the City of New York.

The greatest rectangle is inscribed in an ellipse, and the greatest ellipse in that rectangle, again the greatest rectangle in that (second) ellipse, and the greatest ellipse in that (second) rectangle, and so on *ad infinitum;* show that the sum of all the inscribed rectangles is equal to the area of the rectangle circumscribed about the given ellipse.

Solution by J. E. SANDERS, Reinersville, Ohio, and A. H. HOLMES, Brunswick, Maine.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, be the equation of the ellipse. Let $2x_1$ be the length of its maximum inscribed rectangle. Then $2y_1=2\sqrt{(a^2b^2-b^2x_1^2)}=$its width, and the area of the rectangle is $4x_1y_1=4bx_1\sqrt{(a^2-x_1^2)}/a=$maximum.

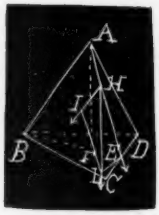
$\therefore 4b\sqrt{(a_2-x_1^2)}-\frac{4bx^2}{\sqrt{(a^2-x_1^2)}}=0$; from which $x_1=\frac{1}{2}\sqrt{2}a=a_1$. Hence, $y_1=\frac{1}{2}\sqrt{2}\,b=b_1$. Hence its area$=4a_1b_1=2ab$. The equation of the ellipse inscribed in this rectangle is $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$. Hence, the length of the maximum rectangle inscribed in this ellipse is $2(\frac{1}{2}\sqrt{2}\,a_1)=2a_2$, its width is $2(\frac{1}{2}\sqrt{2}\,b_1)=2b_2$, and its area is $4a_2b_2=2a_1b_1=ab$. By induction, the area of the nth rectangle$=\frac{1}{2}$ of the area of the $(n-1)$th rectangle, or $a_nb_n=\frac{1}{2}a_{n-1}b_{n-1}$. Hence, the sum of all the rectangles is $S=2ab+ab+\frac{1}{2}ab+\frac{1}{4}ab+\frac{1}{8}ab+\ldots$ *ad infinitum*$=4ab$, which is the area of the rectangle circumscribing the original ellipse.

Also solved by G. B. M. Zerr, J. Scheffer, and G. W. Greenwood.

MECHANICS.

193. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Three light smoothly jointed rods stand like a tripod—the three edges of a regular tetrahedron. A rectangular board, weight w, stands on this like an easel. Find the thrust on the rod which does not touch the easel.



Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Let $AB=a$ be one of the equal rods jointed at A; H the center of gravity of the board, weight w; HL the direction of the weight w. Resolve HL into the components HE along the median AE, and HI parallel to AB. Then HI is the thrust required. Now $AE=BE=\frac{1}{2}a\sqrt{3}$. Draw AF perpendicular to BE. Then $BF=\frac{2}{3}BE=\frac{1}{3}a\sqrt{3}$. $EF=\frac{1}{6}a\sqrt{3}$.

$\cos BAE=\cos IHE=\frac{1}{3}\sqrt{3}$, $\sin IHE=\frac{1}{3}\sqrt{6}$.

$\sin IHL=\frac{1}{3}\sqrt{3}$, $\cos IHL=\frac{1}{3}\sqrt{6}$.

$\sin EHL=\frac{1}{3}$, $\cos EHL=\frac{2}{3}\sqrt{2}$.

$HI:HL=\sin ILH:\sin HIL=\sin EHL:\sin IHE$. $\therefore HI:w=\frac{1}{3}:\frac{1}{3}\sqrt{6}$ or $HI=w/\sqrt{6}=w\sqrt{6}/6$; $HE:w=\sin IHL:\sin HIL$; $HE:w=\frac{1}{3}\sqrt{3}:\frac{1}{3}\sqrt{6}$, or $HE=w/\sqrt{2}=w\sqrt{2}/2$.

Also solved by G. W. Greenwood.

AVERAGE AND PROBABILITY.

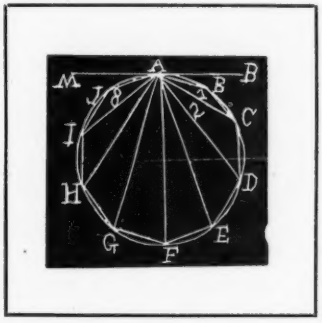
163. Proposed by R. D. CARMICHAEL, Anniston, Ala.

In a regular n-gon a triangle is formed by taking three vertices at random. What is the mean value of the triangle?

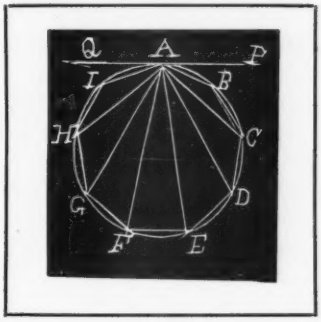
Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Whether n be even or odd, there can be formed, at any vertex, as B, or C, or D, etc., by joining it with A and any other vertex, $(n-2)$ triangles. Since there are n vertices, the total number of triangles $=n(n-2)$.

I. n even. For A the sum of the areas is the area of the polygon $=\frac{1}{2}nr^2\sin(2\pi/n)$; for B and J combined $nr^2\sin(2\pi/n)=$sum of areas; for C and I, the sum is $2nr^2\sin(2\pi/n)-4(n-2)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$; for D and H, $3nr^2\sin(2\pi/n)-4nr^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)-4(n-4)r^2\sin(\pi/n)\times\sin(2\pi/n)\sin(3\pi/n)$. For the next pair of vertices the sum is



n even.



n odd.

$$4n^2\sin(2\pi/n)-4(n+2)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-4(n-2)r^2\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)$$
$$-4(n-6)r^2\sin(\pi/n)\sin(3\pi/n)\sin(4\pi/n).$$

The sum of the areas of the triangles for $\frac{n-2}{2}$th, the $\frac{n}{2}$th and the $\frac{n+2}{2}$th vertices are equal, and their combined sum for the three vertices is

$$\frac{3(n-2)}{4}r^2\sin(2\pi/n)-6(2n-8)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-6(2n-12)r^2\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)$$
$$-6(2n-16)r^2\sin(\pi/n)\sin(3\pi/n)\sin(4\pi/n)$$
$$-\ldots-6(4)\,r^2\sin(\pi/n)\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi.$$

The total sum for all is

$$[\tfrac{1}{2}n+n(1+2+3+\ldots+\frac{n-2}{2})+\frac{n(n-2)}{4}]r^2\sin(2\pi/n)$$
$$-[4r^2(n-2+n+n+2+n+4+\ldots+2n-8)$$
$$+2r^2(2n-8)]\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-[4r^2(n-4+n-2+n+n+2+\ldots+2n-12)$$
$$+2r^2(2n-12)]\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)-\text{etc.},$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-3r^2\sin(\pi/n)[(n-2)(n-4)\sin(\pi/n)\sin(2\pi/n)$$
$$+(n-4)(n-6)\sin(2\pi/n)\sin(3\pi/n)+(n-6)(n-8)\sin(3\pi/n)\sin(4\pi/n)+\ldots$$
$$+8\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi]$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-24r^2[\cos(\pi/n)\cos(2\pi/n)+3\cos(2\pi/n)\cos(3\pi/n)$$
$$+6\cos(3\pi/n)\cos(4\pi/n)+10\cos(4\pi/n)\cos(5\pi/n)+\ldots$$
$$+\frac{(n-2)(n-4)}{8}\cos\frac{n-4}{2n}\pi\cos\frac{n-2}{2n}\pi]\sin(\pi/n)$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-12r^2\sin(\pi/n)\cos(\pi/n)(1+3+6+10+\ldots+\frac{(n-2)(n-4)}{8}]$$
$$-12r^2\sin(\pi/n)[\cos(3\pi/n)+3\cos(5\pi/n)+6\cos(7\pi/n)+\ldots$$
$$+\frac{(n-2)(n-4)}{8}\cos\frac{n-3}{n}\pi]=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-\tfrac{1}{8}n(n-2)(n-4)r^2\sin(2\pi/n)$$
$$+\frac{3n[(n-2)\cos(2\pi/n)-(n-4)]r^2\sin(2\pi/n)}{8[\sin(\pi/n)]^2}$$
$$=\frac{r^2n\{(3n-2)[\sin(\pi/n)]^2-1\}\cos(\pi/n)}{2\sin(\pi/n)}.$$

$$\therefore \text{Average area is } \frac{r^2\cos(\pi/n)}{2(n-2)\sin(\pi/n)}\{[3n-2][\sin(\pi/n)]^2-1\}.$$

II. n odd. By a similar process we easily get for the total sum

$$[\tfrac{1}{2}nr^2+nr^2(1+2+3+4+\ldots+\frac{n-1}{2})]\sin(2\pi/n)$$
$$-4r^2(n-2+n+n+2+n+4+\ldots+2n-7)\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-4r^2(n-4+n-2+n+n+2+\ldots+2n-11)\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)-\text{etc.},$$
$$=\frac{n^3+3n}{8}r^2\sin(2\pi/n)-3r^2\sin(\pi/n)[(n-1)(n-3)\sin(\pi/n)\sin(2\pi/n)$$
$$+(n-3)(n-5)\sin(2\pi/n)\sin(3n/\pi)+(n-5)(n-7)\sin(3\pi/n)\sin(5\pi/n)+\ldots$$
$$+8\sin\frac{n-3}{2n}\pi\sin\frac{n-1}{2n}\pi]$$
$$=\frac{n^3+3n}{8}r^2\sin(2\pi/n)-24r^2\sin(\pi/n)[\cos(\pi/2n)\cos(3\pi/n)$$
$$+3\cos(3\pi/n)\cos(5\pi/n)+6\cos(5\pi/n)\cos(7\pi/n)+\ldots$$
$$+\frac{(n-1)(n-3)}{8}\cos\frac{n-4}{2n}\pi\cos\frac{n-2}{2n}\pi]=\frac{n^3+3n}{8}r^2\sin(2\pi/n)$$
$$-12r^2\sin(\pi/n)\cos(\pi/n)[1+3+6+10+\ldots+\frac{(n-1)(n-3)}{8}]$$
$$-12r^2\sin(\pi/n)[\cos(2\pi/n)+3\cos(4\pi/n)+6\cos(6\pi/n+\ldots$$
$$+\frac{(n-1)(n-3)}{8}\cos\frac{(n-3)\pi}{n}]=\frac{n^3+3n}{8}r^2\sin(2\pi/n)$$
$$-\tfrac{1}{8}(n+1)(n-1)(n-3)r^2\sin(2\pi/n)+\frac{3r^2}{2\sin(\pi/n)}$$
$$-\frac{[3(n^2-2n-3)\sin(2\pi/n)-3(n^2-1)\sin(2\pi/n)\cos(2\pi/n)]r^2}{8[\sin(\pi/n)]^2}$$
$$=\frac{r^2}{4\sin(\pi/n)}\{(4n+3-3n^2)[\sin(\pi/n)]^2\cos(\pi/n)+6+6(n+1)\cos(\pi/n)\}.$$

The average area is

$$\frac{r^2}{4n(n-2)\sin(\pi/n)}\{6+6(n+1)\cos(\pi/n)-(3n^2-4n-3)\cos(\pi/n)[\sin(\pi/n)]^2\}.$$

For the foregoing solutions the following explanation is necessary.

$$2\sin\frac{(p-q)\pi}{n}\sin\frac{(p-q+1)\pi}{n}=\cos(\pi/n)+\cos\frac{(2p-2q+1)\pi}{n},$$

$$2\cos\frac{(p-q)\pi}{2n}\cos\frac{(p-q+2)\pi}{2\pi}=\cos(\pi/n)+\cos\frac{(p-q+1)\pi}{n}.$$

Let $C=\cos3\theta+3\cos5\theta+6\cos7\theta+\ldots+\frac{(n-2)(n-4)}{8}\cos(n-3)\theta,$

$$S=i\sin 3\theta+3i\sin 5\theta+6i\sin 7\theta+\ldots+\frac{(n-2)(n-4)}{8}i\sin(n-3)\theta,$$

where $\theta=\pi/n$, $i=\sqrt{-1}$.

$$\therefore C+S=\cos 3\theta+i\sin 3\theta+3(\cos 5\theta+i\sin 5\theta)+6(\cos 7\theta+i\sin 7\theta)+\ldots$$

$$+\frac{(n-2)(n-4)}{8}[\cos(n-3)\theta+i\sin(n-3)\theta]=(\cos\theta+i\sin\theta)^3+3(\cos\theta+i\sin\theta)^5$$

$$+6(\cos\theta+i\sin\theta)^7+\ldots+\frac{(n-2)(n-4)}{8}(\cos\theta+i\sin\theta)^{n-3}=e^{3i\theta}+3e^{5i\theta}+6e^{7i\theta}+\ldots$$

$$+\frac{(n-2)(n-4)}{8}e^{(n-3)i\theta}=y^3[1+3y^2+6y^4+10y^6+\ldots+\frac{(n-2)(n-4)}{8}y^{n-6}]$$

$$=\frac{y^3}{(1-y^2)^3}-\frac{(n-)(n-4)y^{n+3}+n(n-2)y^{n-1}-2n(n-4)y^{n+1}}{8(1-y^2)^3}.$$

Putting $y=e^{i\theta}=\cos\theta+i\sin\theta$, and equating C=the rational part, we get

$$C=\frac{2n(n-4)\sin(2\pi/n)-n(n-2)\sin(4\pi/n)}{64[\sin(\pi/n)]^3}.$$

MISCELLANEOUS.

164. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the number of real roots of the equation $100\sin x=x$, and show the largest root is approximately 96.10. Find $\tan 39^\circ$ to three places of decimals. How many real roots of $\tan x=1/x^2$ lie between 0 and 2π?

Solution by HENRY HEATON, Belfield, N. D.

(1). The equation may be written $x/\sin x=100$, As x varies from 0 to $\frac{1}{2}\pi$, $x/\sin x$ takes the successive values between 1 and $\frac{1}{2}\pi$. Hence the equation has no real root in the first quadrant. As x varies from $\frac{1}{2}\pi$ to π, $x/\sin x$ takes all the successive values from $\frac{1}{2}\pi$ to ∞. Hence the equation has one real root in the second quadrant. In the third and fourth quadrants $x/\sin x$ is negative. Hence there can be no real root. At every round as x passes through the first and second quadrant $x/\sin x$ varies from ∞ to a minimum value then back to ∞. This minimum value is in the first quadrant when $x=\tan x$. After the first round as long as $x<100$ there are two real positive roots to every round. Hence there can be no real positive root when $x>31\pi$, and the whole number of real positive roots is 31. The number of real negotive roots is evidently the same with the same numerical values. Hence the whole number of real roots is 62. The largest real root evidently occurs between $x=30\frac{1}{2}\pi$ and 31π. Put $x=y+30\frac{1}{2}\pi$. Then $\sin(y+30\frac{1}{2}\pi)=\frac{1}{100}(y+30\frac{1}{2}\pi)$.

$$\therefore \cos y=.958185+\tfrac{1}{100}y.$$

If $\cos y=.95818$, $y=16^\circ\ 38'=.2903^{(r)}$, a first approximate value of y. If $\cos y=.95818+.002903=.96108$, $y=16^\circ\ 2'=.27983^{(r)}$, a second approximation.

If $\cos y = .95818 + .002798 = .96098$, $y = 16^\circ\ 3\frac{1}{2}' = .28027^{(r)}$, a third approximation. If $\cos y = .95818 + .00281 = .96099$, $y = 16^\circ\ 3\frac{3}{8}' = .28023^{(r)}$, a fourth approximation. Whence $x = 96.098$, nearly.

(2). From the well known length of the side of the inscribed decagon we have $\sin 18^\circ = \cos 72^\circ = \frac{1}{4}(\sqrt{5}-1)$.

$$\text{Hence } \cos 36^\circ = \sqrt{\frac{1+\frac{1}{4}(\sqrt{5}-1)}{2}} = \tfrac{1}{4}\sqrt{(6+2\sqrt{5})},$$

$$\text{and } \sin 36^\circ = \sqrt{\frac{1-\frac{1}{4}(\sqrt{5}-1)}{2}} = \tfrac{1}{4}\sqrt{(10-2\sqrt{5})}.$$

$$\text{Hence } \cot 36^\circ = \tan 54^\circ = \sqrt{\frac{6+2\sqrt{5}}{10-2\sqrt{5}}} = \sqrt{(1+\tfrac{2}{5}\sqrt{5})},$$

$$\tan 15^\circ = \frac{1-\cos 30^\circ}{\sin 30^\circ} = 2-\sqrt{3}.$$

$$\therefore \tan 39^\circ = \tan(54^\circ - 15^\circ) = \frac{\sqrt{(1+\frac{2}{5}\sqrt{5})} - 2 + \sqrt{3}}{1+(2-\sqrt{3})\sqrt{(1+\frac{2}{5}\sqrt{5})}}$$

$$= \frac{\sqrt{(5+2\sqrt{5})} - 2\sqrt{5} + \sqrt{15}}{\sqrt{5} + 2\sqrt{(5+\sqrt{5})} - \sqrt{(15+6\sqrt{5})}}.$$

This can be readily computed to any desired degree of accuracy.

(3). As x varies from 0 to $\frac{1}{2}\pi$, $x^2\tan x$ passes through all successive values from 0 to ∞. Hence, it has one value at which $x^2\tan x = 1$. The same is true as x varies from π to $3\pi/2$. But in the second and fourth quadrants all values of $x^2\tan x$ are negative. Hence the equation has no real roots in those quadrants. Hence the number of real roots between 0 and 2π is 2.

Also solved by G. B. M. Zerr and A. H. Holmes. Dr. Zerr finds $\tan 39^\circ = .809785$ by means of $\tan 30^\circ$ and $\tan 9^\circ$. In (3), he finds $x = 51^\circ\ 17\frac{1}{2}'$ and $185^\circ\ 27'\ 10''$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

279. Proposed by THEODORE L. DE LAND, Treasury Department, Washington, D. C.

The United States Panama Canal Bonds were issued, to date August 1, 1906, and will mature on August 1, 1936; and they bear interest at the rate of 2% per annum, payable quarterly, on the first day of November, 1906, and the first day of February, May, and August, 1907, and so on for each succeeding quarter, until the bonds mature, when the principal will be paid at par with the last quarter's interest. The coupon bonds of this loan were quoted on the New York Stock Exchange, at 10.30 a. m., on December 17, 1906, at $103\frac{3}{4}$ bid and $104\frac{3}{4}$ asked.

Required: The rate of interest per annum, payable quarterly, an investor would *realize* if he purchased the Panama bonds on December 17, 1906, and could reinvest his interest income, quarterly, at the *realized* rate.

GEOMETRY.

311. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Dallas High School, Dallas, Texas.

Triangle ABC is obtuse-angled at C; x, y, z are squares on the sides AC, CB, BA; MN and QR are lines joining adjacent sides of x, z and y, z. The common chord of the circles on MN and QR as diameters passes through C and the mid-point of NR.

312. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania, Philadelphia, Pa.

A variable circle passes through a fixed point and is tangent to a given circle. If a diameter of the first circle passes through the fixed point find the locus of its other extremity.

CALCULUS.

235. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

The latitude of a place and two circles parallel to the horizon being given, to determine the declination of a heavenly body whose apparent time of passage from one circle to the other shall be a minimum.

236. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve the partial differential equation, $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial u}{\partial x}$.

MECHANICS.

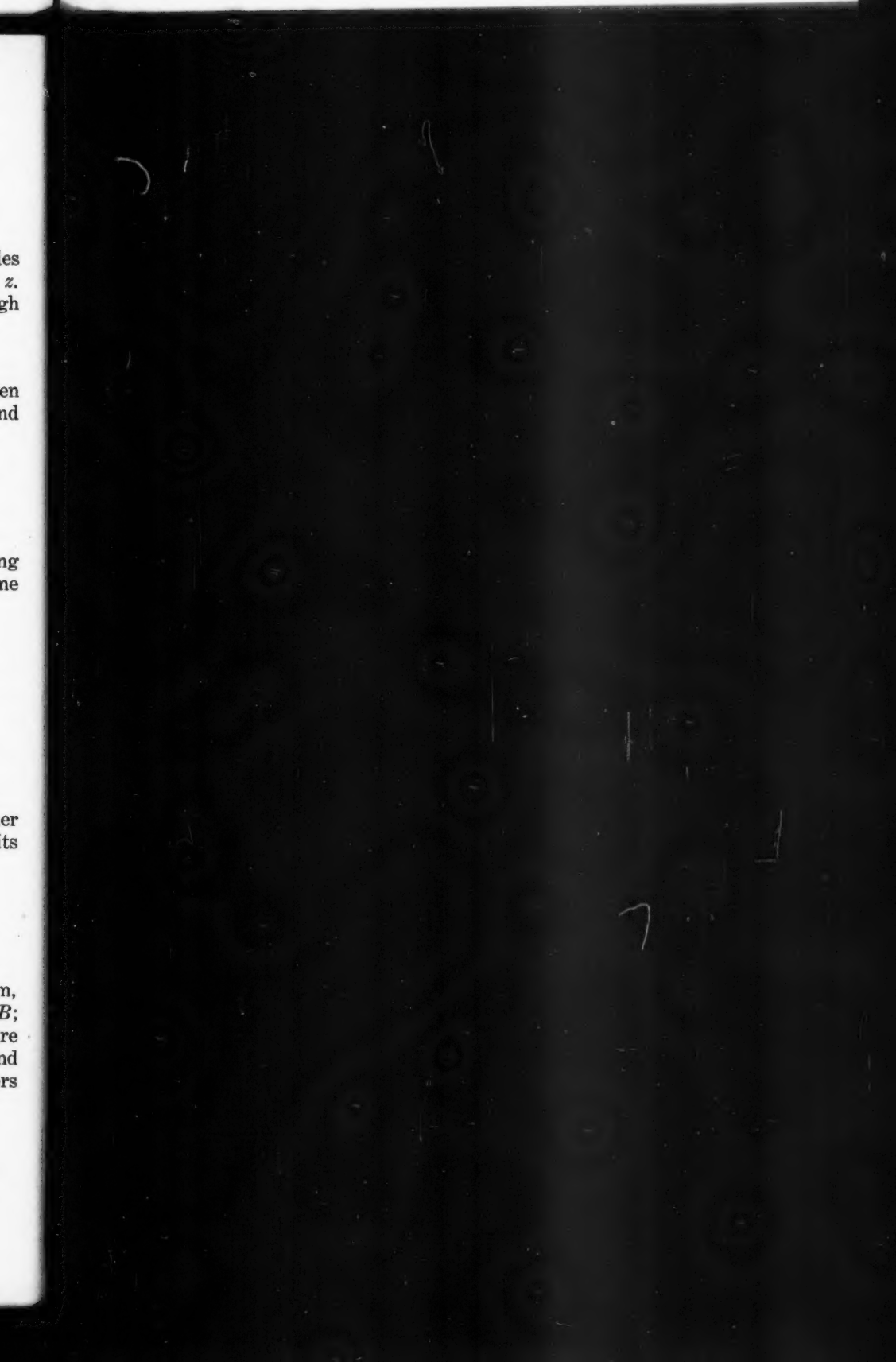
199. Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.

A sphere of water, radius $\frac{1}{4000}$, the earth's radius, is brought together by mutual attractions of particles from a state of infinite diffusion. Find its temperature owing to the amount of work done by these forces.

AVERAGE AND PROBABILITY.

186. Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.

An urn contains $n=100$ balls; $a=25$ balls are stamped, at random, with the letter A; $b=30$ balls are stamped, at random, with the letter B; $c=40$ balls are stamped, at random, with the letter C; $d=50$ balls are stamped, at random, with the letter D. One ball is drawn at random; find the chance it has on it no letter, the letter A, or B, or C, or D, or the letters AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, or $ABCD$.



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ON A SET OF FOUR LINEAR ASSOCIATIVE ALGEBRAIC UNITS.

By S. A. COREY, Hiteman, Iowa.

Dr. Dickson has called attention in the November MONTHLY to a very remarkable set of non-associative algebraic units, and notes that among the many other similar (though usually associative) units, the best known are Hamilton's famous unit vectors. i, j, k, which on account of being perfectly isotropic when applied to Euclidean space, have been of much use to scientists in simplifying the language of analysis as applied to the physical sciences. In this connection it may be of interest to notice very briefly another somewhat remarkable set of "units," in terms of which, when interpreted in a certain manner, scalars, vectors, and quaternions may all alike be written.

We shall begin, as we may, by defining these "units" by a "multiplication table." This is virtually what Hamilton did when he defined his "units" in such a way as to admit of their bearing the interpretation he desired to place on them as unit vectors. Let, then, H, I, J, K, be these four "units" of which the following is the "multiplication table:"

	H	I	J	K
H	H	I	H	$-I$
I	H	I	$-H$	I
J	J	$-K$	J	K
K	$-J$	K	J	K

Further assume that multiplication of these "units" with ± 1 is always commutative, *e. g.* $H(-J)=-HJ$.

Then will multiplication always be associative; for, taking these "units" in sets of three, thus,

$$H.IJ=-HH=-H, \quad HI.J=IJ=-H, \quad J.HK=-JI=K, \quad JH.K=JK=K,$$

and so on until all possible permutations of the four "units" taken three at